NORTH SYDNEY BOYS' HIGH SCHOOL

2008 HSC Course Assessment Task 2

MATHEMATICS

General instructions

- Working time – 60 minutes.
- Write in the booklet provided.
- Each new question is to be started on a new booklet.
- Write using blue or black pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets within this paper and hand to examination supervisors.

Class teacher (please ✔)

- ◯ Mr Fletcher
- ◯ Mr Lam
- ◯ Mr Lowe
- ◯ Mr Ireland
- ◯ Mr Trenwith
- ◯ Mr Rezcallah
- ◯ Mr Weiss

STUDENT NUMBER:

Marker's use only.

QUESTION	MARKS
1	/12
2	/12
3	/11
4	/13
5	/6
Total	/54
Total (%)	/100

Question 1 (12 Marks) Commence a **new** booklet. **Marks**

(a) i. Find $\displaystyle\int 6x^7\,dx$. **2**

ii. Evaluate $\displaystyle\int_0^1 5x^4 + 3x + 1\,dx$. **2**

(b) Find $\displaystyle\int \frac{x^3 + x^2}{2x}\,dx$. **2**

(c) i. Differentiate $y = \left(2x^5 - 1\right)^3$. **2**

ii. Hence or otherwise, find $\displaystyle\int 10x^4\left(2x^5 - 1\right)^2\,dx$. **2**

(d) Find the *exact* value of k if $\displaystyle\int_2^k 3x^2\,dx = 50$. **2**

Question 2 (12 Marks) Commence a **new** booklet.

(a) i. Evaluate $\displaystyle\int_{-a}^{a} x^5\,dx$. **2**

ii. Find the area bounded by the curve $y = x^5$ between the lines $x = -a$, $x = a$ and the x axis. **2**

(b) By sketching the curve $y = \sqrt{9 - x^2}$, hence or otherwise evaluate $\displaystyle\int_0^3 \sqrt{9 - x^2}dx$ as an exact value. **3**

(c) Given $s''(t) = 2t^2$, $s'(2) = 1$ and $s(1) = 2$, find:

i. $s'(t)$ **2**

ii. $s(3)$ **3**

Question 3 (11 Marks) Commence a **new** booklet. **Marks**

(a) Find the area of the region between the curves **4**

$$\begin{cases} y = x^2 \\ y = 3x + 4 \end{cases}$$

(b) Find the volume of solid of revolution when the region bounded by the curve $y = x^2 + 1$, the y axis, the lines $y = 2$ & $y = 5$, rotated about the y axis. **3**

(c) Find an approximation to $\displaystyle\int_0^1 2^x\,dx$ by using Simpson's Rule with 5 function values. **4**

Question 4 (13 Marks) Commence a **new** booklet.

(a) The probability that a particular man lives to the age of 75 is $\frac{4}{5}$ and the probability that his wife will live to 75 is $\frac{6}{7}$. By drawing a tree diagram or otherwise, find:

i. Only the man will live to 75. **1**

ii. Both will live to 75. **2**

iii. At least one of them will live to 75. **2**

(b) In a bag with 20 marbles, seven are red, nine are gold & four are blue. One marble is taken from the bag and not replaced, then a second is taken out.

Find the probability of choosing:

i. Two red marbles. **2**

ii. Marbles of a different colour **2**

(c) If an integer x between 1 and 100 (inclusive) is chosen at random, find the probability of the number being:

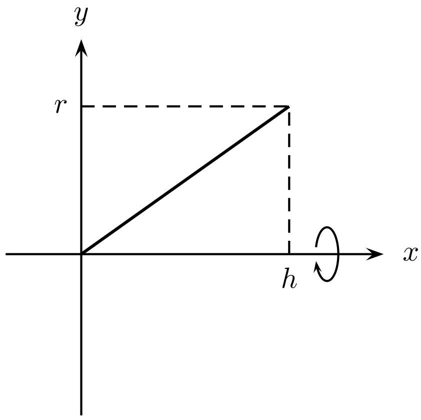
i. Less than 50 or a multiple of 5. **2**

ii. Being a multiple of 9 but not a multiple of 12. **2**

Question 5 (6 Marks) Commence a **new** booklet. **Marks**

(a) i. Find the equation of a line passing through the points $(0, 0)$ and (h, r). **1**

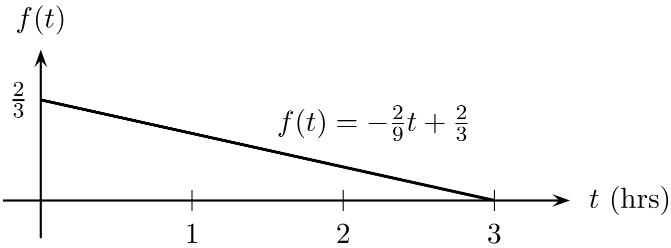
ii. Using the diagram below & by rotating the given line about the x axis, derive the formula for the volume of a cone with radius r and height h. **2**



(b) A certain telecommunications company offers "untimed" international calls but disconnects the call after 3 hours.

The probability of a call duration between $a < t < b$ hours using the function is

$$P(a < t < b) = \int_a^b f(t)\, dt$$



Using this, find the probability of a call lasting

i. Between 60 & 90 minutes. **2**

ii. *Exactly* 1.5 hours. **1**

End of paper.

Solutions

Question 1

(a) i. (2 marks)

$$\int 6x^7\,dx = \frac{3}{4}x^8 + C$$

ii. (2 marks)

$$\begin{aligned}&\int_0^1 5x^4 + 3x + 1\,dx\\ &= \left[x^5 + \frac{3}{2}x^2 + x\right]_0^1\\ &= 1 + \frac{3}{2} + 1 = \frac{7}{2}\end{aligned}$$

iii. (2 marks)

$$\begin{aligned}\int_2^k 3x^2 &= 50\\ \left[x^3\right]_2^k &= 50\\ k^3 - 2^3 &= 50\\ k^3 &= 58\\ k &= 58^{1/3}\end{aligned}$$

(b) (2 marks)

$$\begin{aligned}\int \frac{x^3 + x^2}{2x}\,dx &= \frac{1}{2}\int \frac{x^3}{x} + \frac{x^2}{x}\,dx\\ &= \frac{1}{2}\int x^2 + x\,dx\\ &= \frac{1}{2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right) + C\\ &= \frac{1}{6}x^3 + \frac{1}{4}x^2 + C\end{aligned}$$

(c) i. (2 marks)

$$\begin{aligned}y &= \left(2x^5 - 1\right)^3\\ &\left.\begin{aligned}y(u) &= u^3 & u(x) &= 2x^5 - 1\\ y'(u) &= 3u^2 & u'(x) &= 10x^4\end{aligned}\right\}\\ y'(x) &= y'(u) \times u'(x)\\ &= 3u^2 \times 10x^4\\ &= 30x^4\left(2x^5 - 1\right)^2\end{aligned}$$

ii. (2 marks)

$$\begin{aligned}&\int 10x^4\left(2x^5 - 1\right)^2\,dx\\ &= \frac{1}{3}\int 30x^4\left(2x^5 - 1\right)^2\,dx\\ &= \frac{1}{3}\left(2x^5 - 1\right)^3 + C\end{aligned}$$

Question 2

(a) i. (2 marks)

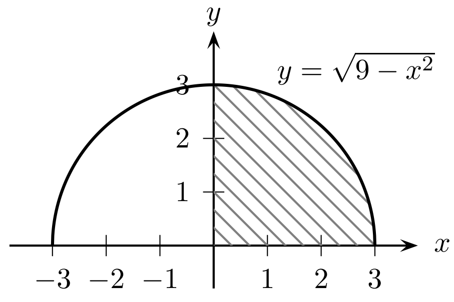
$$\begin{aligned}\int_{-a}^a x^5\,dx &= \left[\frac{1}{6}x^6\right]_{-a}^a\\ &= \frac{1}{6}\left(\cancel{a^6} - \cancel{(-a)^6}\right)\\ &= 0\end{aligned}$$

ii. (2 marks)
Since the integral is 0, then the area from $x = -a$ to $x = 0$ must be equal to the area from $x = 0$ to $x = a$.

$$\begin{aligned}A &= 2\int_0^a x^5\,dx\\ &= 2\left[\frac{1}{6}x^6\right]_0^a\\ &= \frac{1}{3}a^6\end{aligned}$$

Both marks to be awarded iff explanation is acceptable.

(b) (3 marks)



The integral $\int_0^3 \sqrt{9 - x^2}\,dx$ is the same as the area of the quarter circle.

$$\begin{aligned}\int_0^3 \sqrt{9 - x^2}\,dx &= \frac{1}{4}\pi \times 3^2\\ &= \frac{9}{4}\pi\end{aligned}$$

(c) i. (2 marks)

$$\begin{aligned} s'(t) &= \int s''(t)\,dt \\ &= \int 2t^2\,dt \\ &= \frac{2}{3}t^3 + C_1 \end{aligned}$$

Using $s'(2) = 1$

$$\begin{aligned} 1 &= \frac{2}{3} \times 2^3 + C_1 \\ &= \frac{16}{3} + C_1 \\ C_1 &= 1 - \frac{16}{3} = -\frac{13}{3} \\ \therefore s'(t) &= \frac{2}{3}t^3 - \frac{13}{3} \end{aligned}$$

ii. (3 marks)

$$\begin{aligned} s(t) &= \int s'(t)\,dt \\ &= \int \frac{2}{3}t^3 - \frac{13}{3}\,dt \\ &= \frac{2}{3}\cdot\frac{1}{4}t^4 - \frac{13}{3}t + C_2 \\ &= \frac{1}{6}t^4 - \frac{13}{3}t + C_2 \end{aligned}$$

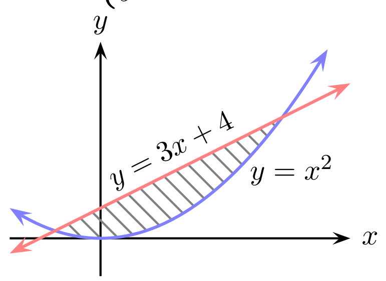
Using $s(1) = 2$,

$$\begin{aligned} 2 &= \frac{1}{6} - \frac{13}{3} + C_2 \\ C_2 &= 2 - \frac{1}{6} + \frac{13}{3} \\ &= \frac{37}{6} \\ \therefore s(t) &= \frac{1}{6}t^4 - \frac{13}{3}t + \frac{37}{6} \\ \therefore s(3) &= \frac{20}{3} \end{aligned}$$

Question 3

(a) (4 marks)

$$\begin{cases} y = x^2 \\ y = 3x + 4 \end{cases}$$



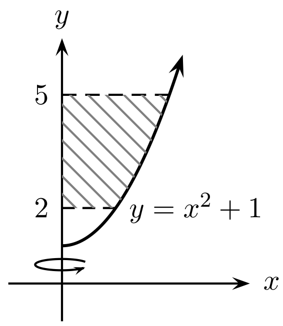
Find the points of intersection between the two curves by equating,

$$\begin{aligned} x^2 &= 3x + 4 \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ \therefore x &= 4, -1 \end{aligned}$$

The area is thus

$$\begin{aligned} A &= \left| \int_{-1}^{4} (3x+4) - x^2\,dx \right| \\ &= \left| \int_{-1}^{4} x^2 - 3x - 4\,dx \right| \\ &= \left| \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x \right]_{-1}^{4} \right| \\ &= \frac{125}{6} \end{aligned}$$

(b) (3 marks)



Changing the subject to x^2:

$$\begin{aligned} y &= x^2 + 1 \\ x^2 &= y - 1 \end{aligned}$$

Integrating,

$$\begin{aligned} V &= \pi \int_2^5 x^2\,dy \\ &= \pi \int_2^5 y - 1\,dy \\ &= \pi \left[\frac{1}{2}y^2 - y \right]_2^5 \\ &= \pi \left(\frac{1}{2}\left(5^2 - 2^2\right) - (5-2) \right) \\ &= \pi \left(\frac{21}{2} - 3 \right) \\ &= \frac{15\pi}{2} \end{aligned}$$

(c) (4 marks)

[2] if the entire table is correct

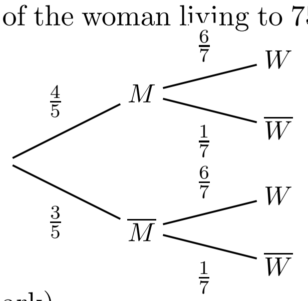
x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
2^x	1	$2^{\frac{1}{4}}$	$2^{\frac{1}{2}}$	$2^{\frac{3}{4}}$	2

Using Simpson's Rule with the 5 function values from the table,*

$$A \approx \frac{h}{3}\left(y_1 + 4y_{\text{even}} + 2y_{\text{odd}} + y_\ell\right)$$
$$= \frac{\frac{1}{4}}{3}\left(1 + 4\left(2^{1/4} + 2^{3/4}\right) + 2\times 2^{1/2} + 2\right)$$
$$= 1.4427 \text{ (4 d.p.)}$$

Question 4

(a) Draw out the probability tree, letting the event of the man living to 75+ be M and the event of the woman living to 75+ be W.



i. (1 mark)

$$P(M) = P(M)\times P(\overline{W}) = \frac{4}{5}\times\frac{1}{7} = \frac{4}{35}$$

ii. (2 marks)

$$P(MW) = P(M)\times P(W)$$
$$= \frac{4}{5}\times\frac{6}{7} = \frac{24}{35}$$

The first mark is for identification of a pair of independent events.

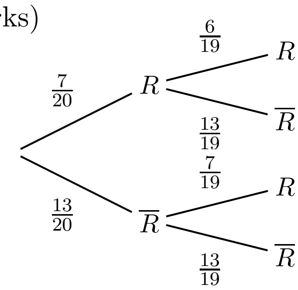
iii. (2 marks)

$$P(\text{at least } M \text{ or } W)$$
$$= 1 - P(\text{neither})$$
$$= 1 - \left(P(\overline{M})P(\overline{W})\right)$$
$$= 1 - \left(\frac{3}{5}\times\frac{1}{7}\right) = \frac{32}{35}$$

The first mark is for identifying the complement.

*Students may use a 4 d.p. approximation of 2^x where 2^x is irrational.

(b) i. (2 marks)



$$P(RR) = \frac{7}{20}\times\frac{6}{19}$$
$$= \frac{21}{190}$$

ii. (2 marks)

$$P(\text{different})$$
$$= 1 - P(\text{same})$$
$$= 1 - \left(P(RR) + P(GG) + P(BB)\right)$$
$$= 1 - \left(\tfrac{21}{190} + \left(\tfrac{9}{20}\times\tfrac{8}{19}\right) + \left(\tfrac{4}{20}\times\tfrac{3}{19}\right)\right)$$
$$= \tfrac{127}{190}$$

(c) A Venn diagram would assist in the calculations. Not drawn here for brevity.

i. (2 marks)
There are 11 multiples of 5 from 50 to 100 inclusive, and 49 numbers less than 50 (1 to 49). Hence

$$P(x < 50 \cup 5|x) = \frac{49+11}{100} = \frac{3}{5}$$

A Venn diagram would assist in the calculations.

ii. (2 marks)
Common multiples of 9 and 12 are 36 & 72. Since there are 11 multiples of 9 between 1 & 100, then there are 9 of them which are not also multiples of 12. Hence

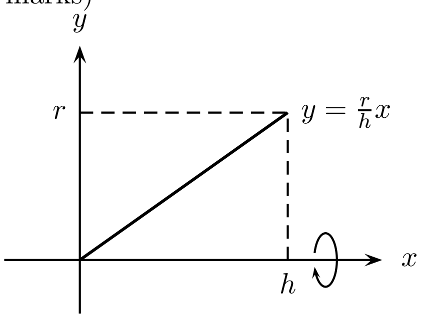
$$P(x|9 \cap 12 \nmid x) = \frac{9}{100}$$

Question 5

(a) i. (1 mark)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{r-0}{h-0} = \frac{r}{h}$$
$$\therefore y = \frac{r}{h}x$$

ii. (2 marks)



$$\begin{aligned} V &= \pi \int_0^h y^2\,dx \\ &= \pi \int_0^h \frac{r^2}{h^2}x^2\,dx \\ &= \frac{\pi r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h \\ &= \frac{\pi r^2}{h^2} \times \frac{1}{3}h^3 \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

(b) i. (2 marks)

$$\begin{aligned} &P(1 < t < 1.5) \\ &= \int_1^{3/2} -\frac{2}{9}t + \frac{2}{3}\,dt \\ &= \left[-\frac{1}{9}t^2 + \frac{2}{3}t\right]_1^{3/2} \\ &= \left(-\frac{1}{9}\left(\frac{9}{4}-1\right) + \frac{2}{3}\left(\frac{3}{2}-1\right)\right) \\ &= \left(-\frac{1}{9}\left(\frac{5}{4}\right) + \frac{2}{3}\times\frac{1}{2}\right) \\ &= \frac{7}{36} \end{aligned}$$

Both marks to be awarded if student is capable of finding the function values at $t = 1$ and $t = \frac{3}{2}$, then using the area of a trapezium to find the proper area.

ii. (1 mark)

$$\begin{aligned} P(t = 1.5) &= P(1.5 < t < 1.5) \\ &= \int_{1.5}^{1.5} f(t)\,dt \\ &= 0 \end{aligned}$$

Highlights the theoretical impossibility of ANY call lasting exactly 1.5 hours under a continuous probability distribution.